





Leveraging Number Lines and Unit Fractions to Build Student Understanding: Insights from a Mixed Methods Study

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Abstract Fractions remain a challenging area of school mathematics at every stage of education, with impacts that extend far beyond the school years. For this study, researchers engaged in classroom-based design research over a 6-year period to investigate effective strategies for teaching fractions with Canadian students. Participants included 86 teachers (representing 12 collaborative research teams spread across 8 school boards) and over 2000 students from Grades 3–10. Quantitative analyses revealed significant pre-post gains in students' fraction knowledge. Qualitative findings revealed some best practices in fractions instruction, including the importance of focusing on unit fractions and number lines to facilitate student sense-making. These findings lead to a detailed discussion of the benefits of (1) focusing on unit fractions as a central construct that allows students to meaningfully work with fractions and make connections across ideas of increasing complexity; (2) leveraging powerful representations as objects-to-think-with that combine concrete and abstract thinking about fractions; and (3) using a design research methodology in the context of collaborative work with teachers.

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Résumé Les fractions demeurent un sujet complexe des mathématiques à l'école, et ce, à chaque étape du processus éducatif, avec des répercussions qui s'étendent bien au-delà des années scolaires. Pour cette étude, les chercheurs se sont engagés dans une recherche conceptuelle réalisée en classe sur une période de six ans afin d'étudier les stratégies qui sont efficaces pour enseigner les fractions aux élèves canadiens. Les participants comprenaient 86 enseignants (représentant 12 équipes de recherche concertée réparties dans 8 conseils scolaires) et plus de 2000 élèves de la 3e à la 10e année. Les analyses quantitatives ont montré des gains importants de connaissance des fractions chez les élèves, avant et après l'étude. Les résultats qualitatifs ont révélé un certain nombre de pratiques exemplaires dans l'enseignement des fractions, notamment l'importance de se concentrer sur les fractions unitaires et de manière visuo-spatiale sur les droites numériques pour faciliter la compréhension des élèves. Ces résultats ont donné lieu à une discussion étoffée sur les avantages de: 1) se concentrer sur les fractions unitaires en tant qu'élément fondamental qui permet aux élèves de travailler de manière pertinente avec les fractions et d'établir des liens entre des idées de complexité croissante; 2) tirer parti de représentations fortes en tant qu'objets de réflexion qui combinent la pensée concrète et abstraite sur les fractions; et 3) recourir à une méthodologie de recherche conceptuelle dans le cadre d'un travail collaboratif avec les enseignants.

Keywords Unit fractions · Spatial reasoning · Visual representations · Number lines · Design research · Mixed methods

Introduction

Vic and Mo are Grade 7 students who have strengths in some areas of mathematics, but both struggle with their understanding of fractions. Both students privately have doubts that even with time and persistence they would be capable of understanding fractions. Outwardly, they make their feelings about fractions known; when their teacher introduces the topic, both students groan, and Mo is overheard to mutter, "I hate fractions, I just don't get it." In today's class, the two students are working together on the problem in Fig. 1, which involves combining 5 partially filled containers, each one-fourth full of liquid, to determine the total amount of liquid altogether.

Using a visual model and focusing on the unit of fourths, we might readily see that the total will be 5 one-fourths (or five-fourths, or 1 and one-fourth) containers worth of liquid. Mo and Vic, however, are stumped by this fairly straight-forward fractions addition problem. Their thinking goes something like this: Mo says "one-fourth, plus one-fourth, plus one-fourth, plus one-fourth, plus one-fourth is a lot of fractions to add together. I'm not very good at fractions, but I know that the top number is called the numerator, and the bottom number is called the denominator, and I know we have to do something

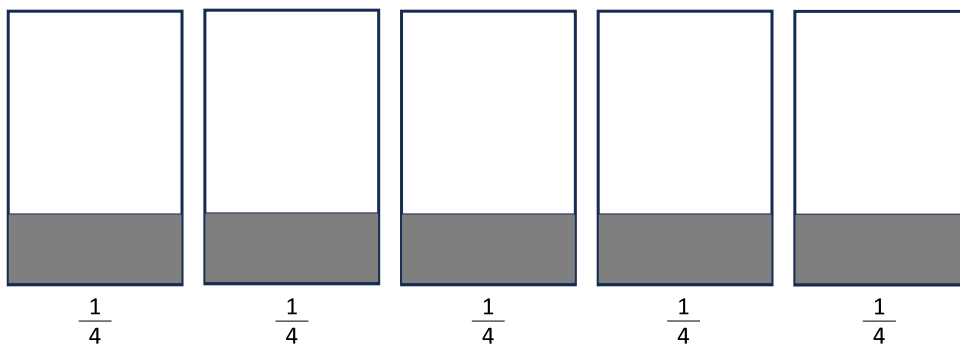


Fig. 1 Visual representation of 5 containers, each one-fourth full of liquid in a question asking students to write a fraction to show how much liquid is in the containers all together

to these numbers.” Vic hesitantly suggests, “this is an adding problem so maybe it makes sense to add up all the numbers on the top and all the numbers on the bottom? That gets us five over twenty.” Mo writes “ $\frac{5}{20}$ ” on the page.

Five-twentieths is actually the same quantity as one-fourth (an equivalent fraction), so in answering “ $\frac{5}{20}$ ”, Mo and Vic have not actually added the fourths together as they would have done to accurately complete the task. They were unable to make sense of the total by combining the fourths, most likely because Vic and Mo have some major—but very common—gaps in their knowledge of fractions. At least two things are happening here. First, they are confused by the symbolic notation of $\frac{1}{4}$ and have imposed some whole number thinking onto the digits in the numerator and denominator positions, treating the 1 and 4 in the fraction as discreet whole number quantities and adding these together. The second major gap in their understanding—closely related to the first and perhaps even more foundational—is that these students have not yet grasped that each fraction represents a single quantity defined not by the numerals 1 and 4, but by the *relationship* between the numerator and denominator.

The scenario of Vic and Mo represents a composite of typical student responses that we observed consistently over the course of a multi-year research study focused on fractions in Ontario, Canada. In fact, 45% of students in the study were unable to answer the question in Fig. 1 correctly on pre-tests. Over a 6-year period, we worked with a total of 86 teachers (representing 12 collaborative research teams spread across 8 school boards) and over 2000 students (Grades 3–10) to generate a comprehensive data set (Yearley & Bruce, 2014; Bruce et al., 2022). We spent an additional 4 years refining teacher resource materials and conducting further analyses of the data. Research on learning fractions has shown that this is challenging content at every stage of education, with impacts far beyond elementary and secondary school (for example, in fields where understanding and accuracy with fractions is critical, such as nursing and pharmacology) (Mackie & Bruce, 2016). Importantly, these well-known difficulties with fractions have been linked to the under-development of children’s overall mathematical thinking and understanding for decades (Charalambous & Pitta-Pantazi, 2007).

As mathematics educators and researchers, we had observed the difficulties students experienced learning fractions firsthand. In designing and conducting the study, we aimed to first learn more about what was causing students to struggle so much in their understanding of fractions. How, for example, did students like Vic and Mo arrive in Grade 7 with such a limited understanding of the meaning of fractions? Our end goal was to work with teachers and students to develop sequenced tasks that addressed these struggles in understanding. In this study, we asked an overarching research question: What instructional strategies support students in the difficult-to-learn area of fractions? Sub-questions included: What foundational ideas underpin fractions understanding that seem to be missing in school math? And how might visual-spatial representations support students in their understanding of fractions? In addition, we asked: To what extent did students’ fraction knowledge improve/not improve over the course of the fraction intervention (i.e., as a potential result of our design research approach to fraction instruction)? To address these questions, we employed a mixed-methods research design, combining both qualitative and quantitative data collection methods and analyses. For our qualitative analyses, we report on our study’s complete database, and for our quantitative analyses, we report on a sub-set of pre-post assessment data.

Theoretical Framing

Many students struggle with fractions learning, a phenomenon particularly well documented in North America and the UK (Bailey et al., 2014; Namkung et al., 2018; Siegler et al., 2011, 2013; Yearley & Bruce, 2014). The National Assessment of Educational Progress and the Programme for International Student Assessment have shown that fractions reasoning is more challenging for students than whole

number reasoning (DeWolf & Vosniadou, 2011, 2015). These challenges are attributed to a range of issues and continue to be debated and studied (Avgerinou & Tolmie, 2020; Lenz et al., 2020; Ong & Chew, 2021). There are a wealth of findings about fractions learning from intervention studies dating back to the 1980s and earlier. More recently, a simple search using ProQuest yields over 500 peer-reviewed published papers in the past 10 years using the terms “fractions learning” and “challenges.” Google Scholar, a relatively broad search tool, yields over 1000 results for the search term “fractions learning” for publications from the past decade, suggesting that this continues to be an area of hot pursuit both theoretically and practically.

What makes understanding fractions so challenging? In part, the answer lies in their mathematical complexity. Fractions have multiple situationally dependent meanings that can involve bidirectional conceptual and procedural thinking (Rittle-Johnson et al., 2015). Depending on the context, a fraction can represent: a part-whole relationship (how much of an object or set is represented in relation to the whole amount); a part-part relationship (quantity relationships that involve ratios); a measurement (distances and equi-distances); magnitude (relative quantities and size); or a mathematical operation (division is inherent in any fraction as all fractions can also be expressed as their quotient, but a fraction can also act as an operator to shrink or grow other quantities, such as $\frac{1}{4}$ or $\frac{5}{4}$ of a litre) (Mosely & Okamoto, 2008). Understanding fractions also involves complex mental processes such as spatial reasoning (Cutting, 2021) and proportional reasoning (Möhring et al., 2016), which are integral to content areas such as probability, measurement, and geometry. Moreover, fractions have been identified in the literature as a necessary foundation for understanding algebra, which itself has been flagged as a gateway to higher mathematics (Brown & Quinn, 2007; Siegler et al., 2013).

An underlying cause for these challenges identified in the literature—and consistently corroborated by our own observations—is that students who struggle with fractions tend to lack an understanding of fractions as quantities. Students frequently apply whole number reasoning to fractions, treating the numerator and denominator as separate whole numbers rather than as one quantity defined by the relationship between these values, as we saw in the vignette at the beginning of this article (Papert, 2010; Empson & Levi, 2011). The nature of this relationship is defined by context; for example, when a fraction represents a part-whole situation, the numerator represents the count and the denominator represents the unit. When working with a fraction as a quotient, on the other hand, the numerator is a dividend of the denominator. This struggle to understand fractions as quantities leads to difficulty with magnitude comparisons (Behr et al., 1992; Bezuk & Bieck, 1993). In our study, and prior to instruction, when we asked students questions such as, “Which is greater, $\frac{2}{3}$ or $\frac{2}{5}$?” they often selected $\frac{2}{5}$ as the greater quantity—attending only to the denominators, and reasoning that 5 is more than 3, so $\frac{2}{5}$ must be greater. Our own observations confirmed a lack of understanding of fractions as quantities in students’ attempts to represent fractions. When we asked Grade 4 (age 10) students to represent the fraction $\frac{4}{10}$, the range of representations revealed a conflation of part-part and part-whole meanings of fractions with an abundance of examples where the students treated the numerator and denominator as whole numbers, with an obscured or unclear relationship. In Fig. 2, we present a photo of a student representation of four-tenths where they made four stacks of ten square tiles. The student knew that the digits four and ten were important, but they were unsure of how the two digits relate to one another and were unable to represent this fraction as a quantity meaningfully.

Instructional choices and curriculum guidelines do not always serve to help students make meaning of fractions. We have considered whether, particularly in North America, the relatively late formal introduction to fractions contributes to the problem. In Asian countries, where mathematics performance is demonstrably higher than in North America, fractions are introduced in the early grades and nearly in parallel with whole numbers (Son et al., 2015). This approach may better capitalize on the “integrated theory of numerical development,” an important work by Siegler and others (Siegler et al., 2011) which posits that we develop our understanding of whole numbers alongside fractions as part of a continuous

Fig. 2 In an exploratory task early in the research cycle, a Grade 4 student attempted to represent $\frac{4}{10}$ by making four stacks of ten square tiles each



integrated number system. In addition, multiple studies suggest that in North America, we typically move too quickly to memorizing procedures involving fractions, without ensuring students have a solid understanding of concepts as basic as fractions-as-quantities. This premature overemphasis on algorithms may actually hinder student understanding in both the short and long term (DeWolf et al., 2015; Gabriel et al., 2013a, b; Hasemann, 1981; Saxe et al., 2013). Students may be capable of following procedures (such as the “invert and multiply” procedure for dividing two fractions) but may generate answers without reason or meaning. This becomes a problem in novel situations, when students struggle with the uncertain application of partially remembered rules, creating challenges in later mathematics (Brown & Quinn, 2006).

In considering how to address these challenges and make fractions more accessible to students, we drew on our experience as researchers and practitioners/curriculum designers in the area of spatial reasoning. We were learning a great deal about the role of spatial reasoning in mathematics education from other branches of our research program. Spatial reasoning involves a range of skills that are mental (and sometimes physical), including “actions” such as perspective taking and visualization; location, orientation, and dimension shifting; transformations and symmetrizing; diagramming and mapping; as well as composing, decomposing, and scaling (Davis et al., 2015; Harris, 2021). Spatial reasoning training/instruction has been shown to lead to gains not only on measures of geometry, but in other mathematical domains, such as number sense and numeration (Bruce et al., 2013b, 2016, 2021; Gunderson et al., 2012; Verdine et al., 2014) including performing calculations (Bates et al., 2022; Mix & Cheng, 2012). Furthermore, spatial reasoning is malleable—it can be improved across ages and genders (Bruce et al., 2013b, 2016, 2021; Uttal et al., 2013). Taken together, these and other studies suggest the importance of attending to and developing children’s spatial reasoning skills (Bruce et al., 2013b, 2016, 2021). However, studies also reveal that geometry and spatial sense are given the least amount of time and attention in classrooms, particularly in early years math (Bruce et al., 2013a, b; Sarama & Clements, 2009), and research findings have had little influence on the re-design of school curriculum to date (Sinclair & Bruce, 2015; Mix & Cheng, 2012). An exception would be the recent release of a revised curriculum in Ontario, Canada, which was influenced by the authors and colleagues of this article.

In our fractions study, we aimed to leverage what we were learning about spatial reasoning in order to increase the effectiveness of instruction in fractions. This is an emerging area of investigation in mathematics education; in spite of the known importance of spatial reasoning in children's math learning, there is a dearth of research on the role of spatial reasoning in fractions learning in particular (Cutting, 2021). A search in Google Scholar for peer-reviewed studies from the past decade using the broad search terms "fractions" and "spatial reasoning" yields only 34 results; in ProQuest, the identical search yields only 2 results. We hope this study will contribute needed insights in this area.

The challenges outlined here have been clearly documented in the literature and were strongly represented in our observations of students in this study. Yet we know that a solid understanding of fractions is crucial for student success in mathematics. A study by Siegler and colleagues (2012), for example, showed that fractions knowledge at age 10 is the strongest predictor of overall success in mathematics at age 16. Challenges with fractions can begin at an early age (Bruce & Flynn, 2011) and persist through schooling and into adulthood, both in daily activity as well as in careers where fractions misconceptions can lead to critical or even fatal errors, such as errors in dosage calculations in medical fields (Ross & Bruce, 2009; Mackie & Bruce, 2016). The stakes are high: a fragile grasp of fractions can damage and/or limit students' school and career prospects.

Research Methodology

A Design Research Approach

Design research (DR) is a methodology we have adapted to our research because it is well-suited to multi-year studies in mathematics education and it involves iterative cycles of designing, testing, and refining learning materials. In this study, we were designing, testing, and refining fractions assessment items and interactive fractions tasks for students, and we were building a developmental learning trajectory to identify the key milestones and sequencing of fractions ideas. Collins and colleagues (2004) explain that this "approach of progressive refinement in design involves putting a first version of a design into the world to see how it works. Then, the design is constantly revised based on experience, until all the bugs are worked out" (p. 18). As Easterday, Lewis, and Gerber (2018) explain: "DR is different from other educational research methodologies in that it studies the effects of previously non-existent interventions on learning. Therefore, it is the approach of choice when current interventions are not sufficient for promoting the desired learning" (p.151).

Importantly, DR has "the dual goals of refining both theory and practice" (Collins et al., 2004, p. 19), which makes it somewhat unique among research methodologies (Easterday et al., 2018). The fractions research was conducted in collaboration with educators in the naturalistic world of the classroom in an effort to bridge research-practice gaps and advance understanding. This bridging of research-practice gaps was described three decades ago by Brown (1992) as a way to accommodate the "messy" and complex nature of classrooms in a way that experimental research is not able to do because there are too many variables to control (Stahl et al., 2019). According to McKenney and Reeves (2020), design research "is powerful for theory building because it privileges ecologically valid studies that embrace the complexity of investigating learning in authentic ... settings" (p. 82).

This study uses both qualitative and quantitative methods in a mixed methods design. Mixed methods are well suited to the complex context and scope of design research because the use of both quantitative and qualitative methods can better shed light on the research problem than either method alone (Creswell, 2015; Creswell & Plano Clark, 2011). While qualitative methods can provide insights with respect to our main research questions (e.g., how do certain instructional strategies support junior and

intermediate students in learning fractions?), quantitative methods can provide an indicator of the degree of impact.

Research Context

Participants

In conducting this study, we worked with over 80 teachers of Grades 3–10 and over 2000 students.

The total population of students and teacher participants in the study is summarized below in Table 1.

Research Activities, Process, and Products

The research was conducted in collaboration with educators in the naturalistic and messy world of the classroom (Stahl et al., 2019) with 6 years of student data collection across Ontario, Canada, and 4 years of additional data analysis and knowledge mobilization.

The participating teachers were all involved in some form of professional learning focused on fractions as part of the research project. Some teachers engaged more intensively as part of a collaborative action research group, and some less intensively as part of field test groups that attended a series of workshops and then implemented pre-designed tasks in their classrooms. The collaborative action research groups (who represented 51% of the participating teachers) met regularly with the researchers, usually for 5 or 6 days over the course of a school year or term to co-plan tasks, implement the tasks and observe student thinking, and refine the tasks in response to student needs and development (see Anderson & Shattuck, 2012; Van den Akker et al., 2006).

These collaborative research meetings followed a general pattern: (i) trialing a task with a group of students; (ii) a debrief of how students responded to the co-designed task; (iii) a refinement of this task or the design of a new task to build on this learning; (iv) an opportunity to work with students in a classroom or in small groups to try the new or refined task; and (v) parting for the day with a commitment by each participant to try the new/refined task with their own students before the next session. At the end of each research cycle (the school year or term in which we were working with a particular group), we had the opportunity to spend 1 or 2 days with the teachers to look over all of the tasks we had designed and trialed with students that year, including any additional tasks that teachers included between sessions. These sessions included focus group sessions and writing sessions in which the team broke out to formally write up the tasks, including any refinements that would help improve functionality.

Over time, this process and the data collected from participating students and teachers formed the foundation for developing a sequence of tasks. For each research cycle, we began the process again with participating research sites (which included some returning and some new teachers), using the most developed sequence of learning materials as a starting point to guide our work together and our

Table 1 Summary of the total study population

Year	# of districts	# of schools	# of teachers	Grades	Ages*	# of students
1	3	11	16	4–7	9–12	461
2	3	6	16	3–8	8–13	368
3	5	18	33	3–9	8–14	856
4	3	12	17	6–10	11–15	410
5	1	1	4	9	14	61
6	2	8	13	7–10	12–15	311
Overall				3–10	8–15	2292

classroom explorations. As a result, the tasks were co-designed and field-tested in iterative cycles by both educators and researchers and carefully sequenced through trials. Over 6 years of progressive refinement and testing, we developed the field-tested sequence and the tasks into a resource we called the Fractions Learning Pathway—a series of foundational concepts and related tasks that emerged as a qualitative product of the design research approach (Van den Akker et al., 2006). We also worked extensively to mobilize knowledge in order to share our research products and findings beyond the project itself, making curriculum support documents and the pathway freely available at www.fractionslearningpathways.ca.

Data Collection and Analysis

Qualitative Data Collection and Analysis

Qualitative data collection included video capture of students in classroom settings, photos and samples of student work, one-to-one and small group interviews with students either in classrooms or in quieter spaces within the schools, as well as the co-designed lesson plans themselves as they evolved through the design research process. Field notes from all debrief and planning meetings with groups of participating teachers were also an important data source. These notes often represented a near-verbatim record of meetings. Throughout, we continually asked ourselves as a group: what are the main building blocks of fractions understanding, and how do these develop in an optimum sequence to support the development of these understandings in students? During our extensive debrief and writing meetings at the end of each research cycle, sequencing and re-sequencing the learning involved a process of disassembling and reassembling the data (Yin, 2011). In the earliest stages of analysis, we coded the data with teachers, using a holistic approach, which involved identifying codes and sorting these into categories (Saldaña, 2016). It was through this process that we identified the clusters of the Fractions Learning Pathway: unit fractions, equivalence and fractions comparison, and operations with fractions. At various stages throughout the process, members of the research team made analytic memos (Saldaña, 2016) to note patterns emerging in the data. In the later stages, we were able to shift to a nested coding strategy (Saldaña, 2016), using key terms that emerged from holistic analysis to look for specific codes and sub-codes. At all stages, researcher utterances were included in the analysis; according to Saldaña (2016), this is appropriate because these interactions were “significant, bidirectional dialogic exchanges of issues and jointly constructed meanings” (p. 17).

Quantitative Data Collection and Analysis (L3)

In this fractions study, we also collected pre-post test data from over 2000 students that were cleaned to arrive at 1502 matched pre-post assessments. The tests consisted of 16 items on average, and each item was scored on a scale from 0 (no response) to 3 (fulsome and accurate response). The scoring was completed by the researchers along with participating educators using a moderated marking approach—a system of grading student responses which are open-ended and require consistent grading standards. We included a process of “second marking”: remarking assessments at random to ensure calibration and ongoing consistency among a network of markers (University of Edinburgh, 2021). Watty et. al. (2014) provide a good description of the many decades of moderated marking practice in what they call “social moderation”:

Social moderation entails multiple assessors judging performances on a specific task, and marking them using a common framework in terms of a common standard. ... Linn (1993) explains that

Table 2 Summary of pre- and post-test scores for 1502 matched pre-post student tests

	Pre-test				Post-test				Effect size (Cohen's <i>d</i>)
	Mean	SD	Min	Max	Mean	SD	Min	Max	
Overall score	7.22	4.77	0	22.00	9.27	4.20	0	18.00	0.46
Unit fractions (part 1)	3.52	2.35	0	9.50	4.25	2.27	0	10.00	0.32
Unit fractions (part 2)	1.69	1.48	0	6.00	1.95	1.55	0	6.00	0.17
Unit fractions (part 3)	1.84	1.34	0	4.00	2.31	1.22	0	4.00	0.37
Comparing fractions	2.45	1.91	0	8.83	3.53	2.22	0	8.00	0.52
Operations with fractions	1.25	1.98	0	12.00	1.49	2.59	0	14.00	0.10

$T(1501) = -21.56, p < .001$, Pre mean = 7.22 (SD = 4.77), Post mean = 9.27 (SD = 4.20)

social moderation depends on the development of a consensus on standards, and on clarifying the performances that satisfactorily meet those standards. A critical aspect of social moderation is staff development and the review of discrepancies in ratings between multiple markers (Linn 1993, p. 467)

Once scoring and reliability of randomized piles of responses were complete, we sorted and cleaned the data thoroughly (ensuring matched pre and post samples of equivalent items and only matched students). The jugs of water task in the opening of this article is an example of a pre-post test item (and was also used for brief qualitative student discussions and interviews).

The 1502 student matched sets ranged from Grades 5 to 10. We used these data to capture a bold indicator of whether there were gains in student test results. We examined test items in three clusters according to the framework established by the Fractions Learning Pathway: (1) unit fractions, (2) equivalence and fractions comparisons, and (3) fractions operations (see Table 2).

Findings

Qualitative Findings: Key Insights

The qualitative data analysis revealed two main findings. First, a focus on unit fractions was a powerful strategy for helping students to understand fractions. Second, leveraging spatial reasoning—in particular visual representations that support meaning-making, such as number lines—provided students with a variety of entry points as well as flexible tools that help them to apply reasonable solutions to even novel situations involving fractions.

Focusing on Unit Fractions as a Foundational Strategy That Builds Understanding

Early in the research process, unit fractions began to emerge as an important foundation for student understanding. A unit fraction is any fraction with a numerator of 1. The fractions $\frac{1}{4}$, $\frac{1}{27}$, and $\frac{1}{105}$ are all unit fractions. Whole numbers are also units. When we add three whole one-digit numbers, the unit is one whole. When we have a number such as 549, we have 5 one-hundred units, 4 tens units, and 9 ones units. The location of the digits matters greatly, just as they do with fractions. Units are the foundation of fractions: when we count fractions, compare fractions, operate with fractions, or measure using fractions—these all involve using unit fractions (Bruce et al., 2013a, b; Carpenter et al., 1993; Charalambous et al., 2010). We experimented extensively with the deliberate use of language, visual

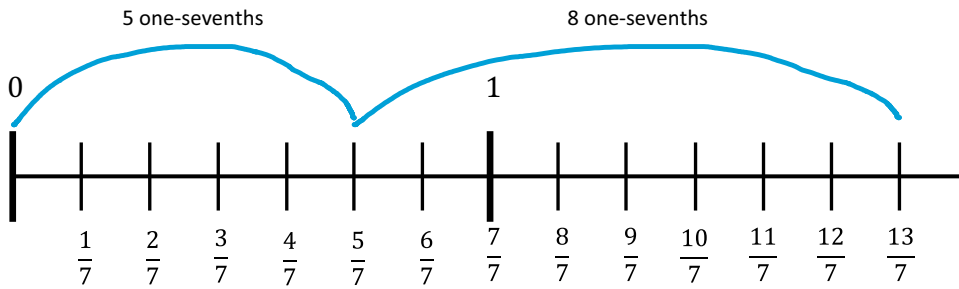


Fig. 3 Composing unit fractions of sevenths on a number line to show how $\frac{5}{7}$ and $\frac{8}{7}$ combine to make a total of $\frac{13}{7}$ or $1\frac{6}{7}$

diagrams, and tasks that focused on unit fractions in our research, and this led to significant gains in student understanding (see quantitative findings below).

We realized that while students had extensive opportunities to count by whole numbers, which helps develop their number sense, including a sense of cardinality, ordinality, and magnitude, the same was not true for fractions. We developed a variety of tasks centred around counting unit fractions. We can count fourths by saying the count (numerator) followed by the unit, just as we might count millimetres or jelly beans or degrees, for example, “1 one-fourth, 2 one-fourths, 3 one-fourths, 4 one-fourths, 5 one-fourths, 6 one-fourths” and so on. (Note that this is different from 1 *and* one-fourth or 2 *and* one-fourth, and in our experience, students had no trouble in distinguishing between the count—an oral strategy—and written fraction amounts.)

In our research, counting by unit fractions proved to be a powerful strategy for reinforcing unit fraction thinking. Once the students were counting by unit fractions, other areas of fractions—such as comparing and adding fractions and, by extension, multiplying and dividing fractions—became less challenging. Using unit fractions, for example, students could add 5 one-seventh units and 8 one-seventh units to arrive at a sum of 13 one-seventh units by adding or even by counting on without much difficulty. They could more easily decompose and re-compose fractions to identify equivalences more easily; for example, they could then identify one whole in seven-sevenths as well as the remaining 6 one-seventh units to see that $\frac{13}{7}$ is the same as $1\frac{6}{7}$ (Fig. 3). Figure 4 uses a number line to visually represent this scenario (composing units of sevenths to represent $1\frac{6}{7}$). When the units are common, adding fractions is an extension of counting unit fractions. But students could also quickly see that adding sevenths and tenths does not work easily because they are not expressed using the same unit. So, finding a common unit became a next obvious action. Teachers across every grade

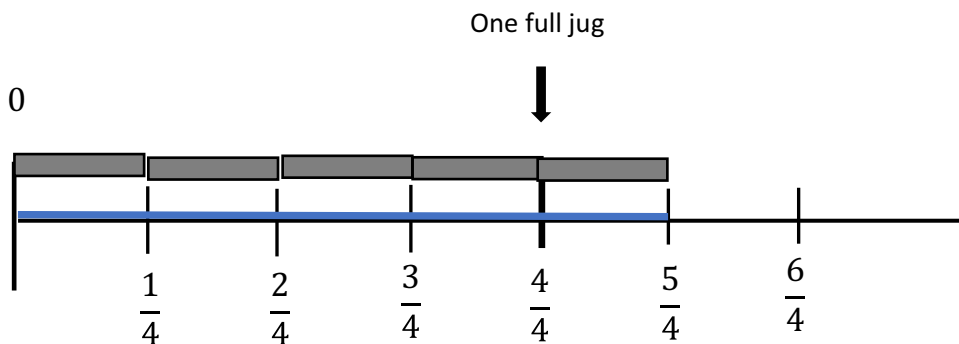
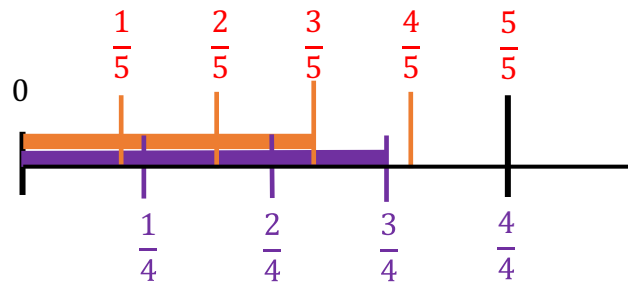


Fig. 4 Composing one-fourth units on the number line to make a quantity of $\frac{5}{4}$ or $1\frac{1}{4}$

Fig. 5 A number line with a length of $\frac{3}{4}$ highlighted in purple and a length of $\frac{3}{5}$ highlighted in orange



consistently and enthusiastically reported that fractions counting tasks had a significant impact on student understanding of fractions overall.

Advantages of Visual-Spatial Reasoning as Powerful Thinking Tools for Fractions Understanding

Powerful thinking tools in mathematics are often visual-spatial in nature. The number line is an example of a visual-spatial object-to-think-with (Papert, 1980), whether virtual or physical, which was particularly powerful in helping students to work with fractions. This linear model helped students with the actions of measuring, addition, and subtraction, but also with understanding concepts such as magnitude, density, unitizing, and comparing quantities. We posit that the abstract and concrete, as well as conceptual and procedural processes, overlap, and we observed this overlap being naturally instantiated in acting with a number line; the overlap is what really makes this a powerful object-to-think-with and -to-act-with. Using a number line to consider fractions requires understanding of, and the related actions for, partitioning (creating equal linear regions on the line), iterating (sequencing these regions repeatedly such as iterations of one-sixth), and disembedding (recognizing the parts *not* in focus as we hold the image of the whole) (Siebert & Gaskin, 2006).

Number Lines in Action

Let us reconsider the task in the vignette at the opening of this article. Students were asked to add 5 one-fourth litres of water together and state how much water they had all together. Using jugs of water may be the most concrete way to join the liquid quantities and see the total quantity. Another option is to represent the quantities of water with a vertical or horizontal linear representation. Using a number line, we can compose 5 one-fourth units in a fairly simple linear fashion: the tool is abstract in its representation of liquid fourths, but we are also operating on the number line concretely.

Suppose now we needed to compare two different quantities: $\frac{3}{4}$ and $\frac{3}{5}$ (Fig. 5). When first considering these denominators, students may assume that the larger denominator must be the fraction with the greater value (fifths are greater than fourths). When we compare these two quantities on a number line, we quickly see that the length of the partitions depends on the unit we are considering: units of fourths are longer than fifths on the same number line. We can also compare the overall lengths of both quantities easily on the number line. Additionally, the number line affords direct comparisons of the same fractions but on different scales. For example, by comparing where $\frac{1}{2}$ is located on different-sized number lines, students are encouraged to focus on the relational aspects of fractions as opposed to absolute distances. The number line uses space (and encourages the use of spatial reasoning) as a way of grounding, and highlighting for students, the inherently proportional/relational meaning of fractions.

As an illustrative example of the emphasis on unit fractions and the power of the number line, we feature here a simple composing-decomposing task from the study. Students were asked to create a

quantity of $1\frac{7}{8}$ (a quantity greater than 1 was selected in order to continue pressing for student flexibility in thinking about fractions). The task required students to find three fractions with a sum of $1\frac{7}{8}$. After discussion in pairs, many students chose to use a number line as their preferred model. Teachers purposely did not provide pre-partitioned number lines, nor rulers, as the very construction of the number line was part of the thinking process for students in considering the magnitude of $1\frac{7}{8}$.

One pair of students partitioned their number line into one-eighth units using part of a pipe cleaner as their benchmark length for each segment. They folded the pipe cleaner into trial lengths until they were satisfied that they had a one-eighth unit to work with for their number line. This pair was able to easily group eighths on their partitioned number line in different ways in order to decompose the total quantity. Equations included simple parsing of the whole and the seven-eighths ($\frac{8}{8} + \frac{3}{8} + \frac{4}{8}$) as well as less obvious compositions such as $\frac{1}{8} + \frac{10}{8} + \frac{4}{8}$. Other pairs in the class used paperfolding for precision or a halving and re-halving strategy. Other pairs came up with mixed-unit solutions, such as $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{5}{10} + \frac{3}{8} = 1\frac{7}{8}$. These students were able to correctly add fractions with different units without converting to a common denominator because they were able to meaningfully compose quantities using unit fractions on a number line.

Quantitative Findings

Analysis of the data from 1502 matched pre-post assessments showed a positive increase in fractions understanding with an overall effect size of 0.46 (using Cohen's d_{av}) calculated as follows:

$$\text{Cohen's } d_{av} = \frac{M_{diff}}{\frac{SD_1 + SD_2}{2}}$$

This effect size corresponds to a moderate effect based on Cohen's guidelines (e.g., see Cohen, 1988). However, according to more recent guidelines, this effect corresponds to a much larger effect (e.g., see Cheung & Slavin, 2016). For this reason, we refer to the overall effects as moderate-to-large in magnitude.

For this article, we further analyzed pre-post test data from Grade 7 and 8 students in particular (ages 11–12). There were over 400 matched pre- and post-tests in this subset ($n=437$). This slice of data was identified because the students in this subset had the opportunity to work through the full sequence of designed tasks from early unit fractions tasks through to multiplication and division with fractions. We also used this sample because (1) we were confident that the professional collaboration between researchers and teachers was consistent with this grouping of grade 7 and 8 teachers (5 days of collaborative work spread throughout the school year), (2) the test items were fully equivalent from pre to post, and (3) the data collection of pre and post student tests were of a consistent interval (pre-testing in early October and post-testing in late May).

Bayesian analyses of the Grade 7 and 8 student results, specifically Bayes factors, provide a statistical way to interpret the degree of change. The statistical model for the results of this study are in favor of there being change is about 583,080 times more likely than a model that predicts (models) no change across time points. This follow-up analysis, combined with the overall results of Table 2, indicates there is clear evidence that students demonstrated significant gains in their fraction knowledge between pre- and post-assessment.

The findings in Table 3 are comparable to the overall data set as reported in Table 2, showing consistency.

Table 3 Fraction performance at pre- and post-test

	<i>N</i>	Mean	SD	95% credible interval	
				Lower	Upper
Pre-test (%)	223	0.568	0.21	0.540	0.596
Post-test (%)	223	0.640	0.23	0.609	0.670

The 95% credible interval provides a measure of estimate uncertainty, i.e., there is a 95% probability that the true parameter (mean) lies in the interval that ranges from the reported lower to upper ends

Discussion

In this study, we reported on our team's 6 years of collaborative research working directly in classrooms with teachers and their students, on the teaching and learning of fractions. As further discussion, we focus on three key insights from the study. The first overarching insight concerns the importance of attending to and further developing children's understanding of unit fractions. A second insight from this study points to the value of using powerful visual-spatial representations that support students in thinking about unit fractions and that can be acted upon to integrate conceptual and procedural thinking. A third key take-away is that the application of a design research approach can be particularly effective for the development of effective learning materials and lessons, but also as an in-depth method for propelling gains in student understanding (as evidenced in the pre-post student results). We end our discussion by underlining the limitations of the study and offering several promising next steps.

Why Unit Fractions are Fundamental to Fractions Teaching and Learning

Unit fractions are the base of all fractions and represent the smallest portion of the fraction we are considering and therefore always have a numerator of 1. All fractions can be broken down or decomposed to unit fractions. We can easily count by unit fractions (1 one-fifth, 2 one-fifths, 3 one-fifths, 4 one-fifths, 5 one-fifths, 6 one-fifths, etc.), as well as add and subtract using unit fractions. Furthermore, unit fractions help us build a sense of quantity in its base form, which can be manipulated—the unit fraction can be composed or iterated to generate greater quantities, and quantities can be decomposed to generate the underlying unit fractions. In our study, students benefitted greatly from participating in counting games in small groups to count up or down using unit fractions. Students created variations of the counting games along the way, where for example, when they reached a whole number in their count, they named the whole number instead of the fraction number. Using number lines for counting is another variation on an oral counting game that leads to a host of variations on the game including “counting up” or “counting down” by unit fractions. In our study, unit fractions understanding was central to helping students more readily perform operations with fractions. Unfortunately, unit fractions have been largely ignored in North American curriculum resources to date. The focus on units when working with fractions fits naturally with other unit thinking across mathematics ideas.

We conclude that the focus on unit fractions—in particular, opportunities to count by unit fractions—helped students because it allowed them to actually capitalize on their strengths in working with whole numbers (such as counting, composing, and decomposing quantities) to support them in internalizing the meaning of fractions as quantities.

Affordances of a Visual-Spatial Emphasis on Number Lines as a Powerful Representation

Powerful representations for fractions thinking includes number lines and related linear models such as ribbons, rods, and strips. We found that linear models have considerable scope—we can return to them over time as fractions concepts become increasingly more complex. In this paper, we provide some examples of the role that number lines played in helping students to make sense of a range of fractions ideas including simple ideas such as equipartitioning, iterating with unit fractions, and composing and decomposing fractions quantities, but also with more complex ideas such as operating with fractions on the number line (Bruce et al., 2022). The main insight however goes beyond the value of using number lines with students for exploration of these fractions ideas: our researcher team and teachers observed consistently that the number line functioned simultaneously as a thinking tool (about fractions ideas) and as a tool for action (taking action on the number line to embody those ideas). Typically, one did not follow the other (thinking then action, or action then thinking) but rather were jointly developed. The physical actions of manipulating the number line (embodied knowing) facilitated complex thinking even with the youngest student participants (see Abrahamson et al., 2020; Radford, 2021).

Affordances of a Design Research Approach with Educators

As described, the fractions study applied a design research approach. This involved working in the complex environment of classrooms with teachers and their students to (1) conduct spontaneous casual interviews with students to help reveal their thinking; (2) trial a range of tasks; (3) reject or refine tasks; (4) test out sequences of tasks in different orders and arrangements; and (5) use different representations in order to optimize and deepen student learning. Working directly with educators provided tremendous insights into student learning and “what works” in classroom contexts. Design research is time-intensive as it is iterative in nature with the goal of building and refining materials or products over time. Each year of the study involved unique populations which gave the researchers opportunities to continually modify the sequences of learning to optimize opportunities for students to develop a deep understanding of fractions and working with fractions. The amount of accumulated in-classroom time in this study was approximated at 300 h, and there were triple this number of hours working with educators beyond the classroom (estimated 900 h). Although not the focus of this paper, the researchers were, over a 6-year period, able to scale up the use of tasks and assessment items to ensure reliability (consistency in the range of student responses) and validity (functioning of the tasks and learning sequences). Overall, the design research approach enabled a slower, more deliberate research structure with opportunities to adjust the scale (number of student participants, number of teachers involved, range of assessment approaches, and intensity of task design and refinements) over time, and overall, to test the learning trajectories that were developed through the process in terms of what types of lesson sequences and tasks enhanced student understanding (and which sequences and tasks did not, and therefore required rejection or refinement and re-testing). The barriers to applying an extended design research approach include the significant time commitment required, sufficient funding and human resources, and being able to find willing educators who have a curious disposition and who are comfortable with engaging in a design research study that takes place largely in their classrooms. We recognize that these are fairly demanding conditions and may not be possible in other contexts.

Limitations

The main limitation of the quantitative results from this study is that we were unable to secure a reliable control group. Researchers made two attempts to establish control groups, one with a delayed treatment model and one with a distinct geography. In both cases, the educators involved in the control sites were

eager to use materials from the study—tasks were precipitously shared by colleagues in treatment groups with educators in the control group. We learned this through follow-up meetings with consultants who worked directly or indirectly with the teachers. The challenges of successfully applying randomized and even quasi-randomized control group methods in educational contexts are well documented (Kelly, 2016; Thomas, 2016). In the end, we were not able to use the control group data as the participants were using enhanced instruction related to shared and publicly available materials from the earlier phases of the study. For these reasons, we turned to Cohen's d and Bayesian analyses as a more reasonable indicator of improvement and impact. However, an important question moving forward is whether the amount of change observed would occur in the absence of this particular intervention: We cannot definitively answer the question of how the change observed in this study compares to “business as usual” fraction instruction. Nonetheless, the gains reported in the present study offer reasons to be optimistic that focusing on unit fractions as central to fractions understanding combined with a liberal use of powerful fractions representations offers a viable model of improving the teaching and learning of fractions moving forward. We even suggest that “business as usual” has led us to the fractions crisis which has been so well documented in the last 30 or more years of research, and that the approach offered in this study marks a concrete positive way forward.

Contributions and Next Steps

This study offers theoretical and practical contributions to the field of mathematics education related to teaching and learning with and about fractions. Theoretically, the study further reinforces the value of visual spatial reasoning through the use of powerful representations such as number lines, which fuse conceptual and procedural knowledge and applications (Niss & Hajgaard, 2019). Within the frame of “objects-to-think-and-act-with,” we see the number line as a dynamic thinking tool which can be partitioned and repartitioned (to reflect the units under consideration) and labelled and re-labelled (to reflect whichever fractions are under consideration). The number line can also be manipulated itself by extending, stretching, or compressing the line, for example. The number line can grow in complexity, illustrating density (there are an infinite number of fractions between any two quantities), and in mathematical utility for multiplication and division with fractions (such as acting on the number line to find two-thirds of one-fourth). We posit that through spatial reasoning, conceptual thinking about fractions and procedural operations collide as students construct understanding with the use of the number line.

As a practical contribution, this study points to a productive shift in pedagogical approaches that focuses squarely on leveraging the underlying structure of unit fractions as a way to unlock understanding of simple fractions through to performing complex operations with fractions. In this study, we developed a pedagogical approach which returns to unit fractions repeatedly during learning sequences as the underpinning idea for understanding and working with fractions. This approach is illustrated in a series of novel and practical field-tested lesson materials for use by educators, including assessment questions with student response samples, engaging tasks with video and explanatory notes, and learning sequences made freely available (www.fractionslearningpathways.ca) and further elaborated and crystallized in a professional resource book for educators (Bruce et al., 2022). The research has also influenced policy in Ontario, with supplementary curriculum resources being generated by the researchers and colleagues. One 28-page PDF commissioned by the Ontario Ministry of Education entitled *Paying Attention to Fractions, K-12*, for example, can be found at <https://www.fractionslearningpathways.ca/pdf/LNSAttentionFractions.pdf>.

In terms of next steps for research, we welcome further use of and testing of the educational materials generated through this research project in different school contexts. But more specifically, we believe the emphasis on teaching using unit fractions and applying the use of unit fractions thinking needs greater attention in curriculum documents, resources, and professional learning programs for teachers.

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Declarations

Competing Interests The authors declare no competing interests.

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